Frequency and damping of the Scissors Mode of a Fermi gas

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We calculate the frequency and damping of the scissors mode in a classical gas as a function of temperature and coupling strength. Our results show good agreement with the main features observed in recent measurements of the scissors mode in an ultracold gas of ^{6}Li atoms. The comparison between theory and experiment involves no fitting parameters and thus allows an identification of non-classical effects at and near the unitarity limit.

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INTRODUCTION

By tuning the atom–atom interaction in a gas of fermions it is possible experimentally to investigate the crossover from a Bose–Einstein condensate (BEC) of molecules to a BCS superfluid. In the crossover region the system is strongly interacting in the sense that $k_F|a| \gg 1$ (unitarity limit), where a is the s-wave scattering length for the atom–atom interaction and $k_F^3 = 3\pi^2 n$, with n being the density of the gas. A theoretical description of this region is challenging, in particular at non-zero temperature [1].

Collective mode experiments have given a wealth of insight into the properties of atomic gases, since they often provide more detailed information than e.g. thermodynamic measurements [2, 3]. In a recent experiment [4] the scissors mode excitation in an elliptical trap was used to characterize the transition between hydrodynamic and collisionless behavior as a function of temperature and scattering length. Since both the frequency and attenuation is affected by the atom–atom interaction, such measurements can give important information on the properties of a fermion system at and near the unitarity limit.

In this Brief Report we calculate the frequency and attenuation of the scissors mode investigated in Ref. [4], assuming the temperature to be sufficiently high that the gas can be treated as being classical. Since the calculation involves no fitting parameters, it can be used to identify non-classical features of the observed frequency shift and damping as functions of temperature and/or interaction strength.

KINETIC THEORY FOR A CLASSICAL GAS

Consider a two-component (for brevity denoted by "spin" with the two values $\sigma =\uparrow,\downarrow$) Fermi gas of atoms with mass m in its normal phase trapped in a potential $V(\mathbf{r}) = m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)/2$. At high temperatures, in the classical regime, the dynamics is described by a semi-classical distribution function $f(\mathbf{r}, \mathbf{p}, t)$, which satisfies the Boltzmann equation. For the scissors mode studied experimentally in [4], the two components of the gas move together, and we need only consider one distribution function $f = f_\uparrow = f_\downarrow$.

To calculate the frequency and damping of the scissors mode, we linearize the Boltzmann equation in terms of a small deviation $\delta f = f - f^0$ from the equilibrium distribution $f^0(\mathbf{r}, \mathbf{p})$ by writing $\delta f(\mathbf{r}, \mathbf{p}, t) = f^0(\mathbf{r}, \mathbf{p})\Phi(\mathbf{r}, \mathbf{p}, t)$. The linearized Boltzmann equation becomes

$$f^0\left(\frac{\partial \Phi}{\partial t} + \dot{\mathbf{r}}\cdot\frac{\partial \Phi}{\partial \mathbf{r}} + \dot{\mathbf{p}}\cdot\frac{\partial \Phi}{\partial \mathbf{p}}\right) = -I[\Phi], \tag{1}$$

where $\dot{\mathbf{r}} = \mathbf{v} = \mathbf{p}/m$ and $\dot{\mathbf{p}} = -\partial V/\partial \mathbf{r}$. The collision integral I is given in Ref. [5] with an interaction described by s-wave scattering involving only particles with opposite spin. The cross section σ is

$$\sigma = \frac{4\pi a^2}{1 + (p_r a/\hbar)^2}, \tag{2}$$

where $\mathbf{p}_r$ is the relative momentum of the scattering particles. The unitarity limit is defined by $|a| \to \infty$.

In the hydrodynamic limit, a scissors mode in the xy plane is characterized by a velocity field $\mathbf{v} \propto \nabla(xy)$ or

$$\mathbf{v} = b(y, x, 0), \tag{3}$$

where b is a constant. To describe this mode we choose an ansatz for δf of the form

$$\Phi(\mathbf{r}, \mathbf{p}, t) = (c_1 xy + c_2 x p_y + c_3 y p_x + c_4 p_x p_y)e^{-i\omega t}, \tag{4}$$

where the c_i are constants and ω is the mode frequency. We insert this ansatz into the linearized Boltzmann equation (1) and take moments by multiplying by any of the terms xy, xp_y, yp_x and $p_x p_y$ appearing in Φ and subsequently integrating over both $\mathbf{r}$ and $\mathbf{p}$. The result is a homogeneous set of four coupled equations for the coefficients $c_1, \ldots, c_4$. The frequencies of the collective modes are determined by the roots of the corresponding determinant which yields the equation

$$\frac{i\omega}{\tau}(\omega^2 - \omega_h^2) + (\omega^2 - \omega_{c1}^2)(\omega^2 - \omega_{c2}^2) = 0. \tag{5}$$

Here

$$\omega_h = \sqrt{\omega_x^2 + \omega_y^2} \tag{6}$$

is the mode frequency in the hydrodynamic limit and

$$\omega_{c1} = \omega_x + \omega_y \text{ and } \omega_{c2} = |\omega_x - \omega_y| \tag{7}$$

the mode frequencies in the collisionless limit [6]. The viscous relaxation rate in (5) is given by

$$\frac{1}{\tau} = \frac{\int d^3r d^3p p_x p_y I[p_x p_y]}{\int d^3r d^3p p_x^2 p_y^2 f^0}. \tag{8}$$

In the classical limit, one obtains [7]

$$\frac{1}{\tau} = \frac{N}{5\pi^2}\frac{m\bar{\omega}^3}{kT}\frac{4\pi a^2}{3}\int_0^\infty dx \frac{x^7}{1+(T/T_a)x^2}e^{-x^2} \tag{9}$$

with $\bar{\omega}^3 = \omega_x\omega_y\omega_z$, while the characteristic temperature T_a is defined by $kT_a = \hbar^2/ma^2$. The integral in (9) comes from averaging over momentum and space the cross section multiplied by an appropriate weight function. In the unitarity limit, (9) gives $\tau^{-1} = 4N\hbar^2\bar{\omega}^3/[15\pi(kT)^2]$. Using (5) and (9) we can now calculate the frequency and damping rate $\Gamma = -\mathrm{Im}\omega$ of the scissors mode as a function of temperature and scattering length. In the hydrodynamic limit $\tau \to 0$, we obtain from (5)

$$\omega = \omega_h - i\tau\frac{2\omega_x^2\omega_y^2}{\omega_x^2+\omega_y^2}. \tag{10}$$

In the collisionless limit $\tau \to \infty$, Eq. (5) yields

$$\omega = \omega_{cj} - \frac{i}{4\tau} \tag{11}$$

with $j = 1, 2$. This result, $\Gamma = 1/4\tau$, for the damping rate in the collisionless limit, obtained by taking moments of the kinetic equation, is in fact exact. This can be seen by treating the collision integral as a perturbation, following the approach used in Ref. [8] for the collisional relaxation of trapped bosons above their condensation temperature.

COMPARISON WITH EXPERIMENT

We now compare our calculated frequency and damping rate of the scissors mode with the recent experiment on trapped ^{6}Li atoms [4]. In Fig. 1, we plot the observed frequency and damping rate as a function of temperature at unitarity, $1/k_Fa = 0$. The scissors mode is excited in the xy plane in a cigar-shaped trap with $\omega_x = 2\pi \times 830$ Hz, $\omega_y = 2\pi \times 415$ Hz, and $\omega_z = 2\pi \times 22$ Hz. We use the relation $\tilde{T} \approx 1.5T/T_F$ with $kT_F = (3N)^{1/3}\hbar\bar{\omega}$ to convert the effective temperature $\tilde{T}$ obtained by fitting the density profile to the real temperature [4, 9]. The calculated frequency and damping from (5) and (9) is plotted as solid black lines. The number of trapped atoms is taken to be $N = 4 \times 10^5$. The dashed black lines in Fig. 1 are the hydrodynamic and collisionless limits given by (10) and (11) respectively. We see that there is good overall agreement between theory and experiment both for the frequency and damping rate as a function of T. For low temperatures, the measured damping is somewhat larger than that obtained for our classical model. Since our model does not include Fermi blocking effects and superfluidity, this discrepancy is not surprising. Note that the theoretical curves contain no fitting parameters.

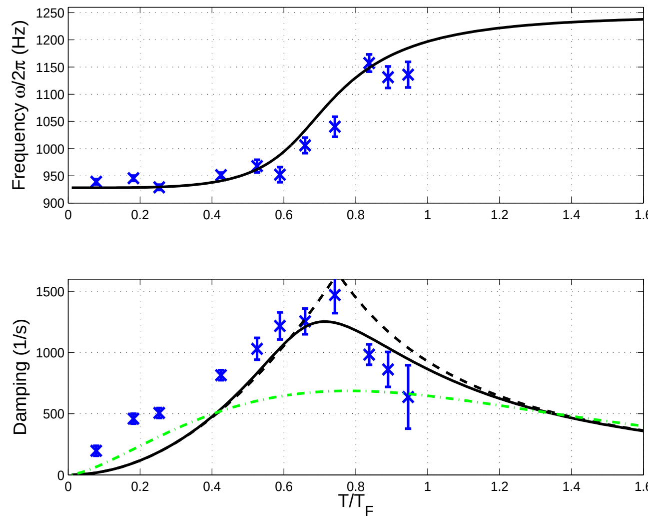

FIG. 1: (Color online) The frequency and damping of the scissors mode in the unitarity limit. The ×'s are experimental values from [4]. The black lines are the theoretical values obtained from the solutions to (5), while the dashed lines indicate the asymptotic results (10) and (11). The green dash-dotted curve is the damping obtained using (12-13) and (15) with $\omega = \omega_h$.

Due to the anharmonicity of the trap, the observed frequency in the collisionless limit is slightly reduced from (11) [10], an effect which is not included in our model, which assumes the trap to be perfectly harmonic.

We also plot in Fig. 1 as a dash-dotted green line the calculated damping rate obtained from hydrodynamics, using the approach described below. When the damping is small, the damping rate is determined by the ratio between the rate of loss of mechanical energy and the energy itself. For the velocity field given by (3), the time average of the mechanical energy, E_{mech}, is equal to

$$\langle E_{\mathrm{mech}}\rangle = \frac{1}{2}\int d^3r mn(\mathbf{r})v^2(\mathbf{r}) = \frac{N}{2}kTb^2(\frac{1}{\omega_x^2}+\frac{1}{\omega_y^2}), \tag{12}$$

since in the classical limit the density is proportional to $\exp(-V(\mathbf{r})/kT)$. The time average of the rate of loss of mechanical energy is $\langle \dot{E}_{\mathrm{mech}}\rangle = -2b^2\int d^3r\eta$, from which the damping rate is obtained as

$$\Gamma = \frac{|\langle \dot{E}_{\mathrm{mech}}\rangle|}{2\langle E_{\mathrm{mech}}\rangle}. \tag{13}$$

In the unitarity limit in the classical regime, a variational

calculation which is accurate to $\lesssim 1\%$ yields [11]

$$\eta = \frac{15}{32\sqrt{\pi}} \frac{(mkT)^{3/2}}{\hbar^2} \tag{14}$$

for the viscosity. This is independent of density and the integration must be cut off close to the surface of the cloud where the gas is no longer hydrodynamic [12]. A cut-off can also be introduced by using the real part of the (complex) dynamical viscosity $\eta(\omega) = \eta/[1 - i\omega\tau_\eta(\mathbf{r})]$ [13] giving

$$\langle \dot{E}_{\mathrm{mech}} \rangle = -2b^2 \int d^3r \frac{\eta}{1 + \omega^2 \tau_\eta(\mathbf{r})^2}, \tag{15}$$

where in the classical limit

$$\tau_\eta(\mathbf{r}) = \frac{\eta}{n(\mathbf{r})kT} = \frac{4.17}{N\bar{\omega}} \left(\frac{kT}{\hbar\bar{\omega}}\right)^2 e^{V(\mathbf{r})/kT}. \tag{16}$$

As can be seen in Fig. 1, this procedure leads to results which are qualitatively correct also in the collisionless regime. It is interesting that the expression (15) yields a damping rate which is exact in the collisionless limit for a trap with cylindrical symmetry ($\omega_x = \omega_y$) provided ω is set equal to its value in the collisionless limit, $\omega = \omega_{c1}$. For a general trap geometry, (15) with $\omega = \omega_{c1}$ yields a damping rate in the collisionless limit which differs from (11) by the factor $8\alpha^2(1+\alpha)^{-2}(1+\alpha^2)^{-1}$ where $\alpha = \omega_y/\omega_x$. For the trap parameters of the experiment in Ref. [4], the factor is ≈ 0.71.

We now examine the transition between hydrodynamic and collisionless dynamics as a function of interaction strength. As in Ref. [4], we define the temperature T_H where the damping is maximum as marking the transition between the hydrodynamic and collisionless regimes. In Fig. 2, we plot as a solid line T_H calculated from (5) and (9) as a function of the interaction strength $k_F a$. The gas can be regarded as hydrodynamic below and collisionless above the line. We also plot the experimentally determined T_H from Ref. [4]. As expected, the transition between collisionless and hydrodynamic behavior occurs at lower temperatures with decreasing interaction strength. There is good agreement between theory and experiment close to the unitarity limit. However, the calculations do not bring out the observed surprisingly steep decrease in T_H with increasing values of $-(k_F a)^{-1}$ on the BCS side of the resonance. This could be due to Fermi blocking effects making the system less hydrodynamic.

The calculations reported here do not take into account Fermi blocking or superfluidity. In the weak-coupling limit ($k_F|a| \ll 1$) it is straightforward to include Fermi blocking effects in the collision integral as described in Ref. [5]. The effect of superfluidity was discussed in Ref. [7] for a uniform superfluid. The viscosity, which is associated with the motion of the normal component, decreases below the transition temperature because of the energy gap in the spectrum of elementary excitations.

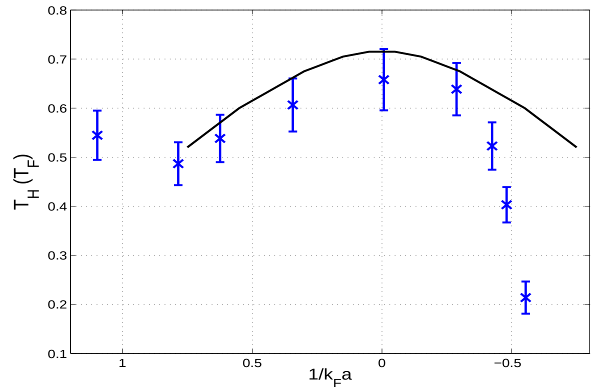

FIG. 2: (Color online) The crossover temperature T_H between hydrodynamic and collisionless dynamics as a function of interaction strength. The $\times$'s are experimental values from [4] and the solid line is obtained as described in the text.

The corresponding change in the viscous relaxation rate is numerically quite small (see Eq. (28) of Ref. [7]). For a trapped superfluid the energy gap depends on position, and it is therefore much more difficult to give a quantitative account of the effects of superfluidity on the mode frequencies.

CONCLUSION

We have analyzed the scissors mode of an interacting Fermi gas in the classical regime. By taking moments of the Boltzmann equation, we have calculated the frequency and damping of the mode both as a function of temperature and interaction strength. The calculation reproduces the main features of the recent experimental findings and can be used to identify non-classical effects for strongly interacting Fermi gases.

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